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Numerical Experiments for Advection Equation

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We propose to combine the Crowley fourth-order scheme and the Gadd scheme for solving the linear advection equation. Two new schemes will be presented: the first is to integrate the Crowley scheme and the Gadd scheme alternately (referred to as New1); the second is to integrate the Crowley scheme twice before we apply the Gadd scheme once (referred to as New2). The new schemes are designed such that no additional restriction is placed on the CFL criterion in an integration. The performance of the new schemes is better than that of the original Crowley or Gadd schemes. It is noted that the amplitude obtained from New2 is more accurate than that from New1 for long waves, but less accurate for short waves. The phase speed calculated from New2 is very close to the real phase speed in most cases tested here, but the phase speed of New1 is faster than the real phase speed. Hence, New2 is a better choice, especially for a model that includes horizontal smoothing to dampen the short waves.

1. INTRODUCTION

Recently, we have been searching for a better scheme for solving the advection equations in the Purdue mesoscale model [1] and have tested several existing methods. In this paper, a new method, which is a combination of the Crowley fourth-order scheme [2] and the Gadd scheme [3, 4], will be presented. This new method performs better than those of the original Crowley or Gadd schemes in the cases we have tested. We have also found that, in comparison with other methods, such as the cubic spline [5], the Warming–Kuttler–Lomax scheme [6], the Roberts–Weiss scheme [9], the fourth-order leapfrog scheme, this new method is quite accurate in solving the advection equation in the solid rotation and the uniform velocity flows. Furthermore, the method is simple and straightforward compared with other third-order or fourth-order schemes.

2. EQUATIONS AND NUMERICAL METHOD

2.a. *Crowley Fourth-Order Scheme (Cr4 Hereafter)*

The two-dimensional advection equation (in a horizontal plane) is

$$\frac{\partial\phi}{\partial t} = -u\frac{\partial\phi}{\partial x} - v\frac{\partial\phi}{\partial y}. \tag{2.1}$$

Following Crowley [2], the difference equation is evaluated in the sense of the alternate-direction (time-splitting) approach, i.e.,

$$\phi^{n+1}_{j,m} = (I-A)(I-B)\,\phi^{n}_{j,m}, \tag{2.2}$$

where A and B represent advection in the x and y directions; i.e.,

$$\phi^{n+1}_{j,m} = (I-A)\,\phi^{+}_{j,m} \tag{2.3}$$

and

$$\phi^{+}_{j,m} = (I-B)\,\phi^{n}_{j,m}, \tag{2.4}$$

where I is the unit matrix, A and B are, thus, one-dimensional operators. The fourth-order, one-dimensional operator becomes

$$\begin{aligned}(B\phi)_{j,m} = {} & \alpha/12[8(\phi_{j+1,m}-\phi_{j-1,m})-(\phi_{j+2,m}-\phi_{j-2,m})] \\ & +\alpha^2/24\,[30\phi_{j,m}-16(\phi_{j+1,m}+\phi_{j-1,m}) \\ & +(\phi_{j+2,m}+\phi_{j-2,m})] \\ & +\alpha^3/12[-2(\phi_{j+1,m}-\phi_{j-1,m}) \\ & +(\phi_{j+2,m}-\phi_{j-2,m})] \\ & -\alpha^4/24[6\phi_{j,m}-4(\phi_{j+1,m}+\phi_{j-1,m}) \\ & +(\phi_{j+2,m}+\phi_{j-2,m})],\end{aligned} \tag{2.5}$$

where the Courant number $\alpha = u\,\Delta t/\Delta x$. Since the calculations have been done separately on the x and y directions, the properties of this scheme can be evaluated separately for each direction. Hence, we will present the eigenvalue for one dimension only. The amplification factor of $(I-B)$ for the component $\exp(ikx)$ is

$$\begin{aligned}f_c = {} & 1-\alpha^2/12(15-16\cos\theta+\cos 2\theta) \\ & +\alpha^4/12(3-4\cos\theta+\cos 2\theta) \\ & -i\{\alpha/6(8\sin\theta-\sin 2\theta) \\ & +\alpha^3/6(-2\sin\theta+\sin 2\theta)\},\end{aligned} \tag{2.6}$$

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TABLE Ia

Amplification Factor as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the Crowley Fourth-Order Scheme (Cr4)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	1.25060	1.11576	1.03685	1.01245	1.00478	1.00206	1.00097	1.00050	1.00027	1.00016	1.00009	1.00006	1.00004	1.00003
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.72260	0.89598	0.97101	0.99066	0.99650	0.99852	0.99930	0.99965	0.99981	0.99989	0.99993	0.99996	0.99997	0.99998
0.80000	0.43360	0.81884	0.95197	0.98481	0.99437	0.99762	0.99889	0.99944	0.99970	0.99983	0.99990	0.99993	0.99996	0.99997
0.70000	0.14660	0.77985	0.94346	0.98235	0.99350	0.99727	0.99873	0.99936	0.99965	0.99980	0.99988	0.99993	0.99995	0.99997
0.60000	0.12640	0.78104	0.94441	0.98278	0.99369	0.99736	0.98877	0.99938	0.99967	0.99981	0.99989	0.99993	0.99995	0.99997
0.50000	0.37500	0.81374	0.95248	0.98534	0.99465	0.99777	0.99896	0.99948	0.99972	0.99984	0.99990	0.99994	0.99996	0.99997
0.40000	0.59040	0.86342	0.96468	0.98911	0.99603	0.99835	0.99923	0.99961	0.99979	0.99988	0.99993	0.99996	0.99997	0.99998
0.30000	0.76540	0.91593	0.97792	0.99318	0.99752	0.99897	0.99952	0.99976	0.99987	0.99993	0.99996	0.99997	0.99998	0.99999
0.20000	0.89440	0.96440	0.98947	0.99675	0.99882	0.99951	0.99977	0.99989	0.99994	0.99996	0.99998	0.99999	0.99999	0.99999
0.10000	0.97340	0.98979	0.99726	0.99915	0.99969	0.99987	0.99994	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000
0.01000	0.99973	0.99990	0.99997	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

where $\theta = k\,\Delta x$ and $i = \sqrt{-1}$. Tables Ia and Ib show the amplification factor $|f_c|$ and the relative phase speed $P = \tan^{-1}\{\mathrm{Im}(f_c)/\mathrm{Re}(f_c)\}/(-\alpha\theta)$ as functions of α and the wavelength $(L = 2\pi/k)$ in Δx units. As pointed out by Crowley [2], this scheme is stable if $|\alpha| \leqslant 1.0$. Both the amplification factor and the relative phase speed approach unity as the wavelength increases.

2b. *Gadd Scheme*

The one-dimensional operator of Gadd's scheme [3, 4] is

$$\begin{aligned}(B\phi)_{j,m} = \alpha[&(1 + 2a/3)/2.0(\phi_{j+1,m} - \phi_{j-1,m}) \\ &- a/6(\phi_{j+2,m} - \phi_{j-2,m})] \\ &- \alpha^2/2[(1 + 4a/3)(\phi_{j+1,m} - 2\phi_{j,m} + \phi_{j-1,m}) \\ &- 4a/3(\phi_{j+2,m} - 2\phi_{j,m} + \phi_{j-2,m})],\end{aligned} \tag{2.7}$$

where $a = 3(1 - \alpha^2)/4$, according to Gadd. The eigenvalue of $(I - B)$ is

$$\begin{aligned}f_g = 1 &- 2\alpha^2 \sin^2(\theta/2)\{1 + (1 - \alpha^2)\sin^2(\theta/2)\} \\ &- 2i\alpha \sin(\theta/2)\cos(\theta/2)\{1 + (1 - \alpha^2)\sin^2(\theta/2)\}.\end{aligned} \tag{2.8}$$

Tables IIa and IIb show the amplification factor and relative phase speed of this scheme. It is also stable for $|\alpha| < 1$. The damping is smaller than in the Lax–Wendroff scheme [8]. The relative phase speed is slightly larger than 1, except for the short waves under a small value of α.

Instead of integrating the x and y directions alternately, Gadd used the two-dimensional Euler scheme to calculate the value at a half-time interval, $\phi^{n+1/2}$; then a central difference scheme was used to calculate the new value at an

TABLE Ib

Relative Phase Speed as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the Crowley Fourth-Order Scheme (Cr4)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	0.00000	−0.37751	−0.81508	1.00406	1.00292	1.00194	1.00129	1.00087	1.00061	1.00043	1.00031	1.00023	1.00018	1.00014
1.00000	0.00000	−0.50000	−1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.00000	−0.66925	0.99081	0.99382	0.99623	0.99768	0.99852	0.99902	0.99933	0.99953	0.99966	0.99975	0.99981	0.99986
0.80000	0.00000	−0.90366	0.97577	0.98588	0.99183	0.99509	0.99692	0.99798	0.98630	0.99904	0.99931	0.99949	0.99962	0.99971
0.70000	0.00000	0.92038	0.95614	0.97673	0.98706	0.99238	0.99527	0.99693	0.99793	0.99856	0.99897	0.99924	0.99943	0.99956
0.60000	0.00000	0.85310	0.93390	0.96704	0.98218	0.98968	0.99366	0.99591	0.99726	0.99809	0.99864	0.99900	0.99925	0.99943
0.50000	0.00000	0.78378	0.91131	0.95750	0.97750	0.98712	0.99215	0.99497	0.99664	0.99767	0.99834	0.99878	0.99909	0.99930
0.40000	0.00000	0.72354	0.89044	0.94876	0.97327	0.98484	0.99082	0.99414	0.99609	0.99730	0.99807	0.99859	0.99895	0.99920
0.30000	0.00000	0.67710	0.87287	0.94137	0.96972	0.98294	0.98972	0.99346	0.99565	0.99700	0.99786	0.99844	0.99883	0.99911
0.20000	0.00000	0.64500	0.85970	0.93579	0.96705	0.98152	0.98890	0.99295	0.99532	0.99677	0.99770	0.99832	0.99875	0.99905
0.10000	0.00000	0.62635	0.85157	0.93232	0.96539	0.98064	0.98839	0.99263	0.99511	0.99663	0.99761	0.99825	0.99870	0.99901
0.01000	0.00000	0.62031	0.84885	0.93116	0.99483	0.98035	0.98822	0.99253	0.99504	0.99659	0.99758	0.99823	0.99868	0.99900

TABLE IIa

Amplification Factor as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the Gadd Scheme (Gm)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	0.91180	0.96132	0.98799	0.99591	0.99842	0.99932	0.99968	0.99983	0.99991	0.99995	0.99997	0.99998	0.99999	0.99999
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.92780	0.97140	0.99196	0.99743	0.99904	0.99960	0.99981	0.99990	0.99995	0.99997	0.99998	0.99999	0.99999	1.00000
0.80000	0.74080	0.90677	0.97522	0.99228	0.99717	0.99882	0.99945	0.99972	0.99985	0.99991	0.99995	0.99997	0.99998	0.99999
0.70000	0.47980	0.83825	0.95918	0.98756	0.99550	0.99813	0.99914	0.99957	0.99977	0.99987	0.99992	0.99995	0.99997	0.99998
0.60000	0.18080	0.79481	0.95009	0.98504	0.99464	0.99779	0.99899	0.99949	0.99973	0.99985	0.99991	0.99994	0.99996	0.99998
0.50000	0.12500	0.79323	0.95043	0.98529	0.99477	0.99786	0.99902	0.99951	0.99974	0.99985	0.99991	0.99994	0.99996	0.99998
0.40000	0.41120	0.83034	0.95908	0.98791	0.99572	0.99825	0.99920	0.99960	0.99979	0.99988	0.99993	0.99996	0.99997	0.99998
0.30000	0.65620	0.88792	0.97251	0.99189	0.99714	0.99883	0.99947	0.99974	0.99986	0.99992	0.99995	0.99997	0.99998	0.99999
0.20000	0.84320	0.94499	0.98627	0.99594	0.99857	0.99942	0.99974	0.99987	0.99993	0.99996	0.99998	0.99999	0.99999	0.99999
0.10000	0.96020	0.98548	0.99633	0.99891	0.99962	0.99984	0.99993	0.99996	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000
0.01000	0.99960	0.99985	0.99996	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

TABLE IIb

Relative Phase Speed as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the Gadd Scheme (Gm)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	0.00000	−0.42882	−0.86044	0.97182	0.98023	0.98546	0.98890	0.99126	0.99294	0.99418	0.99512	0.99585	0.99643	0.99689
1.00000	0.00000	−0.50000	−1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.00000	−0.61527	1.03032	1.02058	1.01497	1.01137	1.00892	1.00717	1.00588	1.00491	1.00416	1.00356	1.00309	1.00270
0.80000	0.00000	−0.79178	1.04809	1.03398	1.02559	1.01992	1.01590	1.01294	1.01071	1.00900	1.00766	1.00660	1.00573	1.00503
0.70000	0.00000	−1.05830	1.05289	1.04084	1.03242	1.02605	1.02120	1.01749	1.01462	1.01237	1.01058	1.00914	1.00797	1.00701
0.60000	0.00000	1.04675	1.04575	1.04221	1.03614	1.03017	1.02510	1.02100	1.01772	1.01509	1.01297	1.01125	1.00984	1.00867
0.50000	0.00000	0.97556	1.02960	1.03951	1.03750	1.03272	1.02787	1.02364	1.02012	1.01725	1.01489	1.01296	1.01136	1.01003
0.40000	0.00000	0.89261	1.00876	1.03440	1.03729	1.03413	1.02974	1.02556	1.02194	1.01891	1.01639	1.01431	1.01257	1.01112
0.30000	0.00000	0.81923	0.98775	1.02848	1.03626	1.03477	1.03095	1.02690	1.02326	1.02014	1.01752	1.01532	1.01349	1.01195
0.20000	0.00000	0.76568	0.97027	1.02319	1.03503	1.03499	1.03167	1.02778	1.02414	1.02098	1.01829	1.01603	1.01413	1.01253
0.10000	0.00000	0.73402	0.95887	1.01958	1.03409	1.03501	1.03205	1.02827	1.02465	1.02147	1.01875	1.01645	1.01451	1.01287
0.01000	0.00000	0.72372	0.95497	1.01832	1.03375	1.03500	1.03216	1.02843	1.02482	1.02163	1.01890	1.01659	1.01464	1.01299

TABLE IIIa

Amplification Factor as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the New Scheme 1 (New1)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	1.06785	1.03566	1.01202	1.00414	1.00160	1.00069	1.00032	1.00017	1.00009	1.00005	1.00003	1.00002	1.00001	1.00001
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.81880	0.93293	0.98143	0.99404	0.99777	0.99906	0.99956	0.99978	0.99988	0.99993	0.99996	0.99997	0.99998	0.99999
0.80000	0.56675	0.86169	0.96353	0.98854	0.99577	0.99822	0.99917	0.99958	0.99977	0.99987	0.99992	0.99995	0.99997	0.99998
0.70000	0.26521	0.80852	0.95129	0.98495	0.99450	0.99770	0.99893	0.99946	0.99971	0.99983	0.99990	0.99994	0.99996	0.99997
0.60000	0.15117	0.78790	0.94725	0.98391	0.99417	0.99758	0.99888	0.99944	0.99970	0.99983	0.99990	0.99994	0.99998	0.99997
0.50000	0.21651	0.80342	0.95146	0.98531	0.99471	0.99781	0.99899	0.99949	0.99973	0.99985	0.99991	0.99994	0.99996	0.99998
0.40000	0.49272	0.84672	0.96188	0.98851	0.99588	0.99830	0.99922	0.99961	0.99979	0.99988	0.99993	0.99996	0.99997	0.99998
0.30000	0.70870	0.90182	0.97521	0.99254	0.99733	0.99890	0.99950	0.99975	0.99986	0.99992	0.99995	0.99997	0.99998	0.99999
0.20000	0.86842	0.95268	0.98787	0.99634	0.99869	0.99946	0.99975	0.99988	0.99993	0.99996	0.99998	0.99999	0.99999	0.99999
0.10000	0.96678	0.98764	0.99679	0.99903	0.99965	0.99986	0.99994	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000
0.01000	0.99967	0.99987	0.99997	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

TABLE IIIb

Relative Phase Speed as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the New Scheme 1 (New1)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	0.00000	−0.40316	−0.83776	0.98794	0.99158	0.99370	0.99509	0.99606	0.99677	0.99731	0.99772	0.99804	0.99830	0.99851
1.00000	0.00000	−0.50000	−1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.00000	−0.64226	1.01056	1.00720	1.00560	1.00452	1.00372	1.00310	1.00261	1.00222	1.00191	1.00166	1.00145	1.00128
0.80000	0.00000	−0.84772	1.01193	1.00993	1.00871	1.00751	1.00641	1.00546	1.00467	1.00402	1.00349	1.00304	1.00268	1.00237
0.70000	0.00000	−0.06896	1.00451	1.00879	1.00974	1.00922	1.00824	1.00721	1.00628	1.00546	1.00477	1.00419	1.00370	1.00329
0.60000	0.00000	0.94993	0.98982	1.00462	1.00916	1.00993	1.00938	1.00846	1.00749	1.00659	1.00580	1.00513	1.00455	1.00405
0.50000	0.00000	0.87967	0.97046	0.99850	1.00750	1.00992	1.01001	1.00930	1.00838	1.00746	1.00662	1.00587	1.00523	1.00467
0.40000	0.00000	0.80808	0.94960	0.99158	1.00528	1.00948	1.01028	1.00985	1.00902	1.00810	1.00723	1.00645	1.00576	1.00516
0.30000	0.00000	0.74816	0.93031	0.98493	1.00299	1.00886	1.01033	1.01018	1.00945	1.00857	1.00769	1.00668	1.00616	1.00553
0.20000	0.00000	0.70534	0.91498	1.97949	1.00104	1.00825	1.01028	1.01036	1.00973	1.00887	1.00800	1.00718	1.00644	1.00579
0.10000	0.00000	0.68018	0.90522	0.97595	0.99974	1.00783	1.01022	1.01045	1.00988	1.00905	1.00818	1.00735	1.00660	1.00594
0.01000	0.00000	0.67201	0.90191	0.97474	0.99929	1.00767	1.01019	1.01048	1.00993	1.00911	1.00824	1.00741	1.00666	1.00599

TABLE IVa

Amplification Factor as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the New Scheme 2 (New2)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.10000	1.12559	1.06170	1.02017	1.00690	1.00266	1.00114	1.00054	1.00028	1.00015	1.00009	1.00005	1.00003	1.00002	1.00001
1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.78539	0.92045	0.97794	0.99291	0.99735	0.99888	0.99947	0.99973	0.99985	0.99992	0.99995	0.99997	0.99998	0.99999
0.80000	0.51835	0.84716	0.95966	0.98730	0.99530	0.99802	0.99908	0.99953	0.99975	0.99986	0.99991	0.99995	0.99997	0.99998
0.70000	0.21766	0.79885	0.94867	0.98408	0.99417	0.99756	0.99887	0.99943	0.99969	0.99982	0.99989	0.99993	0.99996	0.99997
0.60000	0.14242	0.78561	0.94630	0.98353	0.99401	0.99750	0.99884	0.99942	0.99969	0.99982	0.99989	0.99993	0.99996	0.99997
0.50000	0.26001	0.80684	0.95180	0.98532	0.99469	0.99780	0.99898	0.99949	0.99972	0.99984	0.99991	0.99994	0.99996	0.99998
0.40000	0.52334	0.85225	0.96281	0.98871	0.99593	0.99832	0.99922	0.99961	0.99979	0.99988	0.99993	0.99996	0.99997	0.99998
0.30000	0.72712	0.90650	0.97611	0.99275	0.99739	0.99892	0.99951	0.99975	0.99987	0.99992	0.99995	0.99997	0.99998	0.99999
0.20000	0.87700	0.95526	0.98840	0.99648	0.99873	0.99948	0.99976	0.99988	0.99994	0.99996	0.99998	0.99999	0.99999	0.99999
0.10000	0.96898	0.98835	0.99695	0.99907	0.99967	0.99986	0.99994	0.99997	0.99998	0.99999	0.99999	1.00000	1.00000	1.00000
0.01000	0.99969	0.99988	0.99997	0.99999	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000

TABLE IVb

Relative Phase Speed as a Function of the Courant Number α and the Wavelength in Grid Lengths (Δ) for the New Scheme 2 (New2)

α	2	3	4	5	6	7	8	9	10	11	12	13	14	15Δ
1.1000	0.00000	−0.39461	−0.83020	0.99331	0.99536	0.99645	0.99716	0.99767	0.99805	0.99835	0.99858	0.99877	0.99893	0.99906
1.00000	0.00000	−0.50000	−1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
0.90000	0.00000	−0.65126	1.00398	1.00274	1.00248	1.00224	1.00199	1.00174	1.00152	1.00132	1.00116	1.00102	1.00090	1.00080
0.80000	0.00000	−0.86637	0.99988	1.00191	1.00309	1.00337	1.00324	1.00297	1.00266	1.00236	1.00210	1.00186	1.00166	1.00148
0.70000	0.00000	0.26082	0.98839	0.99810	1.00218	1.00360	1.00392	1.00379	1.00349	1.00316	1.00284	1.00254	1.00228	1.00205
0.60000	0.00000	0.91765	0.97118	0.99210	1.00017	1.00318	1.00414	1.00428	1.00408	1.00376	1.00342	1.00308	1.00278	1.00251
0.50000	0.00000	0.84771	0.95074	0.98484	0.99750	1.00232	1.00406	1.00453	1.00447	1.00419	1.00386	1.00351	1.00318	1.00288
0.40000	0.00000	0.77990	0.92988	0.97730	0.99461	1.00127	1.00379	1.00461	1.00471	1.00450	1.00418	1.00383	1.00349	1.00317
0.30000	0.00000	0.72448	0.91116	0.97041	0.99190	1.00022	1.00346	1.00461	1.00485	1.00471	1.00441	1.00407	1.00372	1.00339
0.20000	0.00000	0.68523	0.89655	0.96493	0.98971	0.99934	1.00315	1.00456	1.00492	1.00484	1.00457	1.00423	1.00388	1.00354
0.10000	0.00000	0.66224	0.88734	0.96141	0.98829	0.99876	1.00294	1.00451	1.00496	1.00491	1.00465	1.00432	1.00397	1.00363
0.01000	0.00000	0.65478	0.88423	0.96021	0.98780	0.99856	1.00287	1.00450	1.00497	1.00493	1.00468	1.00435	1.00400	1.00366

$n+1$ time step, ϕ^{n+1} [3]. (This scheme will hereafter be referred to as G78). In 1979, Gadd [4] proposed an alternate operator for the calculation of ϕ^{n+1} by bringing additional gridpoint values (cross-space terms) into the second step of the calculation (referred to hereafter as G79).

2c. *Proposed New Schemes*

Tables Ib and IIb show that the relative phase speed of the Crowley fourth-order scheme is smaller than one, but the relative phase speed of the Gadd scheme is faster than unity for a wavelength longer than four or five grids. Both schemes are stable for α being less than unity. Hence, if we combine the Crowley fourth-order scheme and the Gadd scheme, we may obtain a better relative phase for waves with wavelength $\leqslant 10$ grid lengths.

Two methods are proposed here: one is to integrate the Crowley scheme and the Gadd scheme alternately. Each cycle includes two time steps (the Crowley scheme is from n to $n+1$ time steps, and the Gadd scheme is from $n+1$ to $n+2$ time steps). For each cycle, the new amplification factor equals the amplification factor of Crowley times the amplification factor of Gadd scheme, and the new relative phase speed is the summation of both schemes. The amplification factor and the relative phase speed per each time step of this scheme (referred to as New1) are shown in Tables IIIa and IIIb.

In the second method we integrate the Crowley scheme two times, before we integrate the Gadd method once. Hence, each cycle involves three time steps (the Crowley scheme is used from n to $n+1$, and from $n+1$ to $n+2$; and the Gadd method is used from $n+2$ to $n+3$). The amplification factor and relative phase speed per each time step of this scheme (referred to as New2) are shown in Tables IVa and IVb. From Tables III and IV we may conclude that New2 is better than New1 for long waves.

Our primary interest is to apply this scheme to multi-dimensional models. Hence, it will be easier to integrate the Gadd scheme (G78) along the x- and y-directions separately, as done by Crowley. (This modified Gadd scheme is referred to as Gm). Stability analysis derived from one dimension, then, is also valid for our two-dimensional cases. We have also tested and found that the performance of Gm is better than G79 under both rotational and uniform flows.

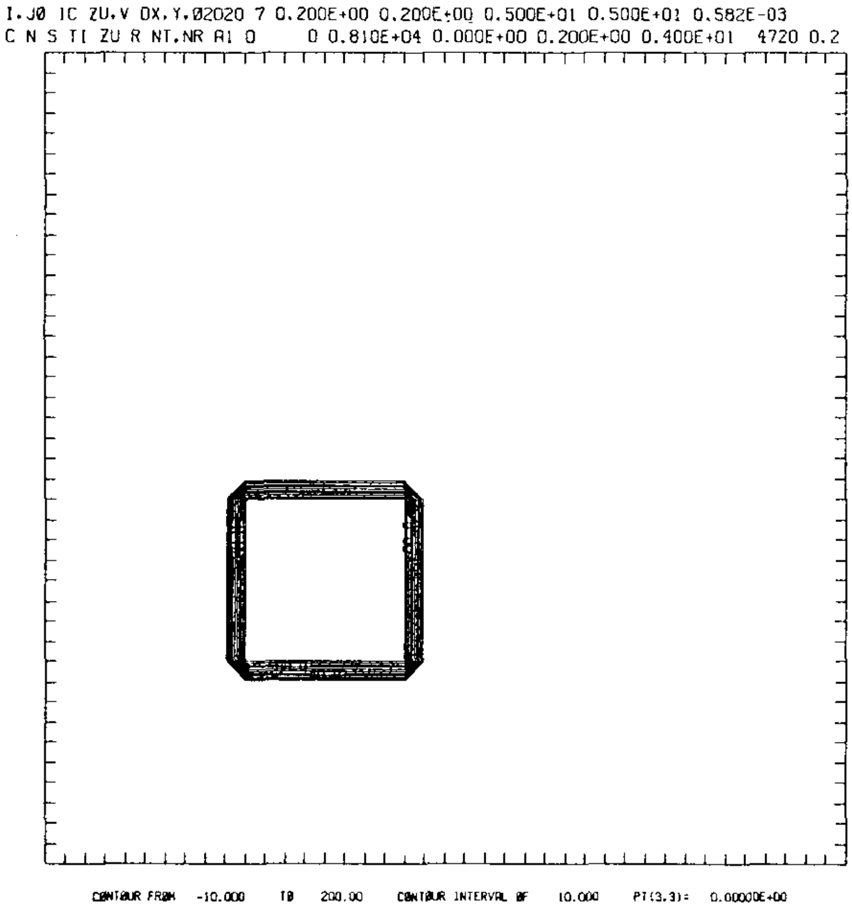

FIG. 1a. Initial square step function.

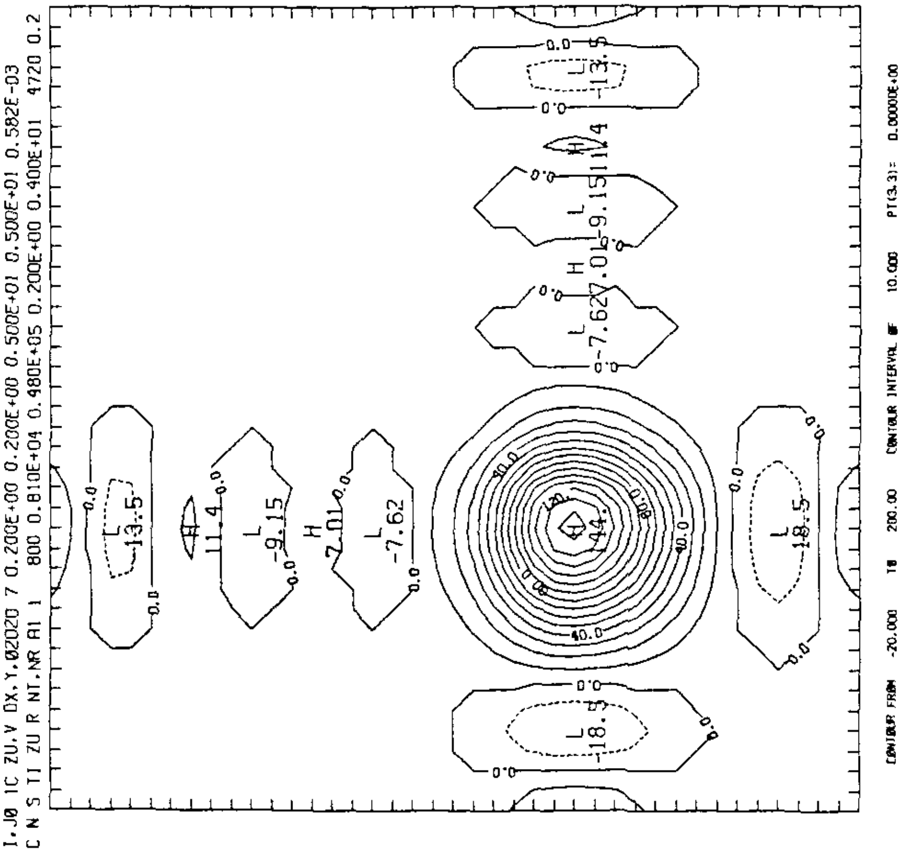

FIG. 1b. Simulation after 800 time steps (4 cycles) using Cr4 with $\alpha = \beta = 0.2$.

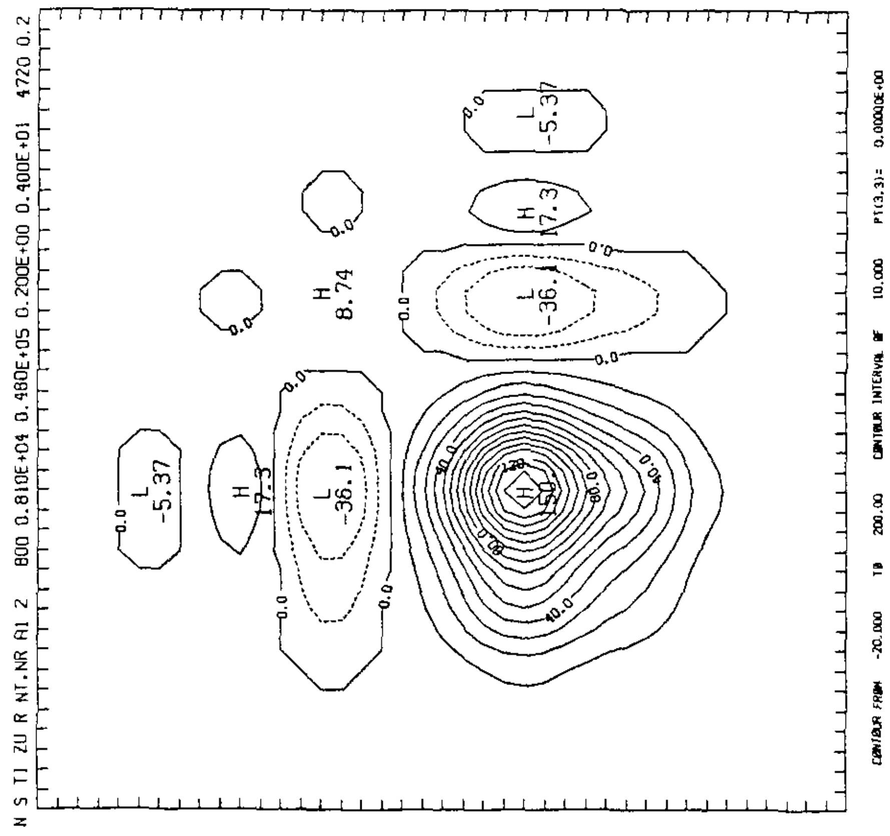

FIG. 1c. Same as 1b but using Cm.

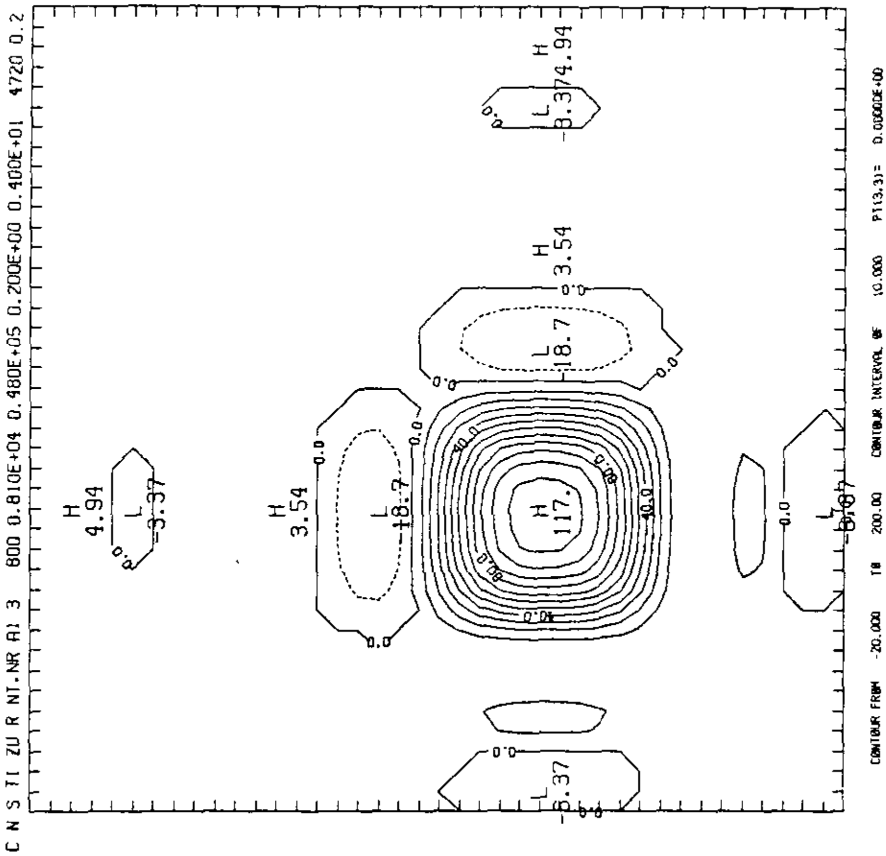

FIG. 1d. Same as 1b but using New1.

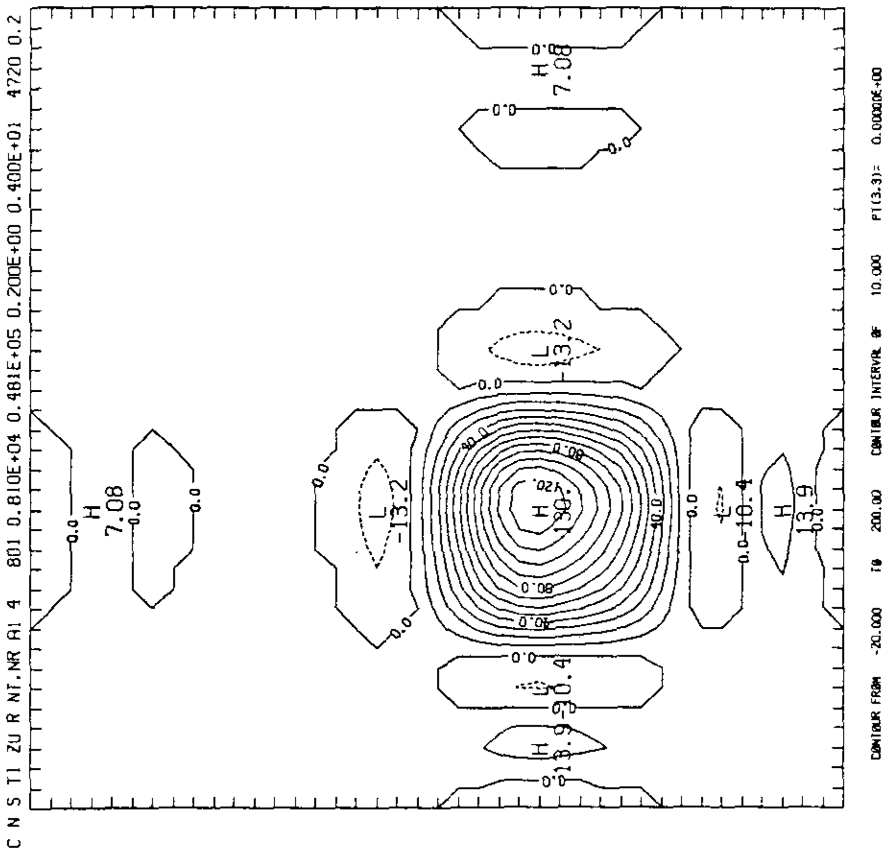

FIG. 1e. Same as 1b but using New2 after 802 time steps.

3. NUMERICAL RESULTS FOR A UNIFORM FLOW

The entire domain consists of 41×41 grids with a cyclic boundary condition. Here, we test a square step function, which is

$$\phi_{j,m} = \begin{cases} 100 & \text{for} \quad 11 \leqslant j \leqslant 19 \text{ and } 11 \leqslant m \leqslant 19 \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

The initial value of $\phi_{j,m}$ is shown in Fig. 1a. The Courant number α $(= u\,\Delta t/\Delta x) = \beta$ $(= v\,\Delta t/\Delta y) = 0.2$. It takes 800 time steps to complete four revolutions. Figure 1b shows the simulation obtained from Cr4 after 800 time steps. The maximum reaches 144, and the minimum reaches -18.5. Furthermore, the tailed perturbations already reach the main body through the periodic boundary. The result of Gm is about the same as Cr4 (Fig. 1c). The result of New1 (Fig. 1d) is much better than Cr4. The square shape is well preserved, even though it advances one grid ahead of the true location. The perturbations generated from New1 are also ahead of the main body. The result obtained from New2 (Fig. 1e) has a larger maximum (130) and minimum (-13.9) in comparison with that obtained from New1 (117, -18.9). The phase speeds of Cr4 and New2 are quite accurate in this case.

4. CONCLUDING REMARKS

From numerical experiments presented in this paper, we may conclude that the numerical schemes, New1 and New2, which are derived by combining the Crowley fourth-order scheme and a modified Gadd schemes, are better than the original Crowley scheme or Gadd scheme. Among these two schemes, New1 usually produces the maximum amplitude more accurately for short waves, and New2 is more accurate for long waves (e.g., "$L \geqslant 5\,\Delta x$"). The phase speed of New2, however, is usually more accurate for most cases after a long integration. Hence, New2 is the better choice among these two, especially for most models that include horizontal smoothing to filter the short waves.

The schemes proposed here are simple. They can be applied in either two- or three-dimensional flows in exactly the same form. Another advantage is that the schemes are stable when $|\alpha| \leqslant 1$ and $|\beta| \leqslant 1$, which is larger than the criterion of the fourth-order leapfrog scheme. The computational time required for New1 and New2 are comparable with that of either the Crowley fourth-order scheme or the Gadd scheme. The New2 scheme has been incorporated into the Purdue mesoscale model to study the vortex shedding behind a large island [9]. The propagation of the lee-vortices is faster than that simulated from the original Crowley scheme.

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